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The holographic RG flow in a field theory on a curved background

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Abstract: As shown by Freedman, Gubser, Pilch and Warner, the RG flow in $\mathcal{N} = 4$ super-Yang-Mills theory broken to an $\mathcal{N} = 1$ theory by the addition of a mass term can be described in terms of a supersymmetric domain wall solution in five-dimensional $\mathcal{N} = 8$ gauged supergravity. The FGPW flow is an example of a holographic RG flow in a field theory on a flat background. Here we put the field theory studied by Freedman, Gubser, Pilch and Warner on a curved AdS_4 background, and we construct the supersymmetric domain wall solution which describes the RG flow in this field theory. This solution is a curved (non-Ricci flat) domain wall solution. This example demonstrates that holographic RG flows in supersymmetric field theories on a curved AdS_4 background can be described in terms of curved supersymmetric domain wall solutions.

Keywords: Renormalization Group, AdS/CFT and dS/CFT Correspondence, Supersymmetry and Duality.

Contents

1. Introduction

The AdS/CFT correspondence [1, 2, 3], together with the domain wall/QFT correspondence [4], states that renormalization group (RG) flows in field theories may be described in terms of domain wall solutions in dual gauged supergravity theories. An example of such a holographic RG flow has been constructed by Freedman, Gubser, Pilch and Warner [5]. This is a flow in $\mathcal{N} = 4$ super-Yang-Mills theory broken to an $\mathcal{N} = 1$ theory by the addition of a mass term for one of the three adjoint chiral superfields [6]. Its dual description is in terms of a supersymmetric domain wall solution in five-dimensional $\mathcal{N} = 8$ gauged supergravity. This domain wall is supported by two non-constant scalar fields, which are in one-to-one correspondence with the fermionic and bosonic mass terms in the dual field theory. The $\mathcal{N} = 2$ version of this solution has been given in [7].

Holographic RG flows have, so far, only been studied in field theories on a flat background. Here we put the field theory studied by Freedman, Gubser, Pilch and Warner on a curved AdS_4 background and we show that the RG flow in this field theory has a dual description in terms of a curved (non-Ricci flat) supersymmetric domain wall solution in five-dimensional gauged supergravity. We construct this curved domain wall solution in the context of $\mathcal{N} = 2$ gauged supergravity in five dimensions. This demonstrates that holographic RG flows in supersymmetric field theories on a curved AdS_4 background can be described in terms of curved supersymmetric domain wall solutions.

In [8] we gave a general recipe for constructing curved supersymmetric domain wall solutions in the context of five-dimensional $\mathcal{N} = 2$ gauged supergravity with vector and hypermultiplets. We also explicitly constructed an example of a curved domain wall solution in a gauged supergravity model with one hypermultiplet. Related work appeared in [9, 10, 11]. In [8] we also discussed the dual description of these curved BPS domain wall solutions in terms of RG flows. We proposed that 1) curved BPS domain wall solutions may provide a dual description of RG flows in field theories on a curved background with AdS_4 curvature and 2) that the curvature on the domain wall may act as an infrared regulator in the dual field theory. Here we provide evidence for 1).

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The curved BPS flow equations describing the curved version of the FGPW flow are complicated and difficult to solve. Actually, even in a flat background there is no known analytic solution to the FGPW flow, making it very difficult to analyze fluctuations and correlation functions far away from the fixed points. We therefore restrict ourselves to constructing the curved domain wall solution in the vicinity of the UV fixed point. We show that the curved BPS domain wall solution is supported by more scalar fields than its flat counterpart, and that the scalar fields (which involve both vector and hyper scalar fields) supporting the curved domain wall solution are in one-to-one correspondence with mass operators in the dual field theory, some of which are induced by putting the field theory on a curved background.

2. A curved version of the FGPW domain wall solution

The flat domain wall solution constructed by Freedman, Gubser, Pilch and Warner is a solution to five-dimensional $\mathcal{N}=8$ gauged supergravity, and it interpolates between two AdS vacua. These two vacua correspond to two RG fixed points in the dual field theory, one in the UV regime and the other one in the IR regime.

The FGPW solution can also be described in the context of $\mathcal{N}=2$ gauged supergravity in five dimensions [7]. In the following, we will construct a curved version of this flat domain wall solution. The five-dimensional $\mathcal{N}=2$ gauged supergravity theories that we consider are in the class constructed in [12], describing the general coupling of vector and hypermultiplets to supergravity.

The flat domain wall solution constructed in [7] arises in a five-dimensional $\mathcal{N}=2$ gauged supergravity theory with scalar manifold

$$\mathcal{M} = \mathrm{O}(1,1) \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times \mathrm{U}(1)}\,. \tag{2.1}$$

The $\mathrm{O}(1,1)$ factor denotes the vector scalar manifold, and it is parametrised by one vector scalar ρ. The metric of this very special manifold is given by $g_{\rho\rho} = 12\rho^{-2}$. The factor $\mathrm{SU}(2,1)/\mathrm{SU}(2)\times\mathrm{U}(1)$, on the other hand, denotes a quaternionic Kähler space which is parametrised by four scalar fields belonging to a hypermultiplet. These four scalar fields are denoted by $q^X = (V,\sigma,\theta,\tau)$ and the associated line element reads $ds^2 = \frac{1}{2}V^{-2}dV^2 + \frac{1}{2}V^{-2}(d\sigma - 2\tau d\theta + 2\theta d\tau)^2 + 2V^{-1}(d\theta^2 + d\tau^2)$.

The flat domain wall solution of FGPW is then obtained by performing a gauging of two compact isometries of the quaternionic Kähler space described above. This gauging results in the following triplet of Killing prepotentials [7] (we refer to [7] for the details)

$$\begin{aligned} P^s = \sqrt{\frac{3}{2}}\Bigg(&\frac{2\theta + V\theta\rho^6 - \theta^3\rho^6 - \sigma\tau\rho^6 - \theta\tau^2\rho^6}{\sqrt{V}\rho^2}, \frac{\sigma\theta\rho^6 + \tau(2 + V\rho^6 - \theta^2\rho^6 - \tau^2\rho^6)}{\sqrt{V}\rho^2},\\ &\frac{4(-V+\theta^2+\tau^2) - \rho^6(1+\sigma^2+V^2+\theta^4+2\theta^2\tau^2+\tau^4-6V(\theta^2+\tau^2))}{4V\rho^2}\Bigg)\,. \end{aligned} \tag{2.2}$$

This triplet of Killing prepotentials may be decomposed into its norm W and into SU(2) phases Q^s according to $P^s = \sqrt{\frac{3}{2}} W Q^s$ with $Q^s Q^s = 1$. The associated W^2 is then given by

$$\begin{aligned} W^2 = \frac{1}{16V^2\rho^4} \big(& 16V[\sigma\theta\rho^6 + \tau(2 + V\rho^6 - \theta^2\rho^6 - \rho^6\tau^2)]^2 + \\ & + 16V[\theta^3\rho^6 + \sigma\rho^6\tau + \theta(-2 - V\rho^6 + \rho^6\tau^2)]^2 + \\ & + [-4(-V + \theta^2 + \tau^2) + \rho^6(1+\sigma^2+V^2+\theta^4+2\theta^2\tau^2+\tau^4-6V(\theta^2+\tau^2))]^2 \big) \,, \end{aligned} \tag{2.3}$$

and the triplet Q^s is given by $Q^s = \sqrt{\frac{2}{3}} P^s/W$.

The flat domain wall solution constructed in [7] is supported by the vector scalar ρ and by the hyper scalar field τ. More precisely, the hyper scalars supporting this flat solution satisfy [7]

$$\sigma = 0 \tag{2.4}$$

as well as

$$\xi^2 = \theta^2 + \tau^2 \,, \qquad \xi = \sqrt{1 - V} \,, \qquad 0 < V \leq 1 \,. \tag{2.5}$$

The condition (2.4) is necessary for the domain wall solution to preserve residual supersymmetry. The particular combination of θ and τ appearing in (2.5) indicates that the flat domain wall solution possesses a U(1) symmetry associated with the existence of a non-anomalous U(1) current for the entire RG flow in the dual field theory [5]. For simplicity, we may therefore set

$$\theta = 0 \tag{2.6}$$

without loss of generality. The UV fixed point is at $\rho = V = 1$, while the IR fixed point is at $\rho = 2^{1/6}$, $V = 3/4$. In the dual field theory (and using the $\mathcal{N} = 1$ component notation of [13]), a non-constant vector scalar ρ corresponds to the addition of a mass term for one of the adjoint scalar fields, $\rho \leftrightarrow m^2 A\bar{A}$, whereas a non-constant hyper scalar field τ corresponds to the associated ($\mathcal{N} = 1$ supersymmetric) fermionic mass term, $\tau \leftrightarrow m\chi\chi$.

We now proceed to construct a curved version of this flat domain wall solution. We write the five-dimensional line element as

$$ds^2 = \mathrm{e}^{2\,\mathrm{U}(r)} g_{mn} dx^m dx^n + dr^2 \,, \tag{2.7}$$

where the metric g_{mn} is taken to be a four-dimensional constant curvature metric satisfying $R_{mn} = 12\Lambda^2 g_{mn}$, and where Λ denotes a real constant. This corresponds to a four-dimensional anti-de Sitter spacetime.

There is a characteristic quantity which enters in the construction of curved supersymmetric domain wall solutions, and which is given by [8, 14]

$$\gamma = \sqrt{1 - 4\Lambda^2(\mathrm{e}^U gW)^{-2}} \,. \tag{2.8}$$

A flat domain wall satisfies $\Lambda = 0$. A curved version will receive corrections to all orders in Λ, as suggested by power expanding γ in Λ. Here we will restrict ourselves to constructing a curved domain wall solution to order Λ^2.

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The curved domain wall solution will be supported by some of the scalar fields ρ, V, σ, θ and τ. As in the flat case, we again set

$$\sigma = 0\,, \qquad \theta = 0\,. \tag{2.9}$$

We will check that the restriction (2.9) is consistent with both the equations of motion and the curved BPS flow equations for the scalar fields. We also write

$$\tau = \sqrt{1 - V + f}\,, \tag{2.10}$$

for convenience. The curved domain wall will thus be supported by the three scalar fields $\rho(r), \tau(r)$ and $f(r)$.

According to the general recipe given in [8] for constructing curved BPS domain wall solutions, we first have to specify a triplet M^s satisfying $M^s M^s = 1$, $M^s Q^s = 0$. In addition, M^s has to be consistent with [8]

$$BM^s = \mp\gamma \frac{\partial_\rho Q^s}{W^{-1}\partial_\rho W}\,, \tag{2.11}$$

where $B = \sqrt{1-\gamma^2} = 2\Lambda(\mathrm{e}^U gW)^{-1}$. We take

$$M^s = \frac{1}{[(Q^2)^2 + (Q^3)^2]}(0, -Q^3, Q^2)\,. \tag{2.12}$$

For domain walls satisfying (2.9) we find that $\partial_\rho Q^s \propto M^s$, so that (2.12) and (2.11) are indeed consistent with one another when subjected to (2.9).

From (2.11) we obtain (we choose the upper sign in the following)

$$B\gamma^{-1} = -\frac{M^s \partial_\rho Q^s}{W^{-1}\partial_\rho W} = 12\alpha^{-1} f(2+f)\sqrt{1-V+f}\sqrt{V}\rho^6\,, \tag{2.13}$$

where

$$\begin{aligned} \alpha = {} & 8(1+f)^2 + 2(2 + f(2+f))(1 + f - 2V)\rho^6 - \\ & - (2 + f(2+f))^2\rho^{12} + 16(1+f)V\rho^{12} - 16V^2\rho^{12}\,. \end{aligned} \tag{2.14}$$

On the other hand, since

$$B\gamma^{-1} = 2\Lambda(\gamma \mathrm{e}^U gW)^{-1}\,, \tag{2.15}$$

we see from (2.13) that whenever the wall is curved ($\Lambda \neq 0$) we have $f \neq 0$ and $\tau \neq 0$ (away from the fixed points).

As mentioned above, we will restrict ourselves to constructing the curved domain wall solution to order Λ^2. To this order we obtain $2\Lambda(\mathrm{e}^U gW)^{-1}$ for the l.h.s. of (2.13).

Since f if of order Λ and higher, we expand the r.h.s. of (2.13) in powers of f. To linear order in f we then obtain

$$f = -\frac{2}{3g}\Lambda \mathrm{e}^{-U_0}\sqrt{\frac{V_0}{1-V_0}}\rho_0^{-4}\left[-1 + (-1 + 2V_0)\rho_0^6\right]\,, \tag{2.16}$$

where we used (2.9). Here the subscript 0 refers to the flat solution. This thus determines f to lowest order in the cosmological constant Λ on the wall.[1] To quadratic order in f, on the other hand, we obtain from (2.13)

$$\begin{aligned}\Lambda \mathrm{e}^{-U} &= -\frac{3g}{2}\frac{\sqrt{1-V}}{\sqrt{V}}\rho^4 \frac{f}{[-1+(-1+2V)\rho^6]} + \\ &\quad + \frac{3g}{4}\frac{\sqrt{V}}{\sqrt{1-V}}\rho^4 \frac{[1+(-3+2V)\rho^6]f^2}{[-1+(-1+2V)\rho^6]^2}\,, \end{aligned} \tag{2.17}$$

where we again used (2.9). Using (2.16) and (2.17), we find that, to quadratic order in Λ, f is given by

$$\begin{aligned} f &= -\frac{2}{3g}\Lambda \mathrm{e}^{-U}\sqrt{\frac{V}{1-V}}\rho^{-4}\left[-1+(-1+2V)\rho^6\right] + \\ &\quad + \frac{2}{9g^2}\Lambda^2 \mathrm{e}^{-2U_0}\frac{V_0^2}{(1-V_0)^2}\rho_0^{-8}\left[1+(-3+2V_0)\rho_0^6\right] \times \\ &\quad \times \left[-1+(-1+2V_0)\rho_0^6\right] + \mathcal{O}(\Lambda^3)\,. \end{aligned} \tag{2.18}$$

Here, U, V and ρ denote the order Λ-corrected expressions, whereas U_0, V_0 and ρ_0 denote the expressions appearing in the flat domain wall solution. The expressions for U, V and ρ will be determined by solving the associated curved BPS equations for these fields. The expression (2.18) for f, on the other hand, should solve the curved BPS equation for this field to order Λ^2. This will indeed turn out to be the case.

Let us now check that the curved domain wall specified by (2.9), (2.12) and (2.18) satisfy the scalar equations of motion given in [8]. The equation of motion for the vector scalar ρ reads (we again take the upper sign)

$$\left(\rho' \partial_\rho W \partial_\rho \Gamma - \gamma' \partial_\rho W\right) + \gamma^2 q^{X\prime}\left(\Sigma_X{}^Y \partial_\rho \partial_Y W + \gamma^{-1}\partial_\rho \partial_X W\right) = 0\,, \tag{2.19}$$

where the prime denotes the derivative with respect to r, and where

$$\begin{aligned} \Gamma^{-2} &= 1 + W^2 \frac{(\partial_\rho Q^s)(\partial_\rho Q^s)}{(\partial_\rho W)(\partial_\rho W)}\,, \\ \Sigma_X{}^Y &= -\gamma \delta_X{}^Y + 4\Lambda\left(\mathrm{e}^U g W\right)^{-1}\varepsilon^{rst} M^r Q^s R^t_X{}^Y\,. \end{aligned} \tag{2.20}$$

Working to order Λ^2, we find that (2.19) contains terms proportional to f^2 and to Λ^2. To arrive at this result, we have made use of the curved BPS flow equations which we will discuss below (cf. (2.22)). We find that the terms proportional to f^2 and to Λ^2 precisely cancel out by virtue of (2.16)! The equations of motion for the hyper scalar fields are given by

$$\begin{aligned} &\gamma^{-2}\rho'\partial_\rho W \partial_Z \Gamma - 4g\gamma W \Sigma_Z{}^Y \partial_Y W - 4gW\partial_Z W + q^{X\prime}\left(\partial_Z \Sigma_X{}^Y\right)\partial_Y W - \Sigma_Z'{}^Y \partial_Y W - \\ &\quad - q^{X\prime}\left(\Sigma_Z{}^Y \partial_X \partial_Y W - \Sigma_X{}^Y \partial_Z \partial_Y W\right) - \rho'\left(\gamma^{-1}\partial_Z \partial_\rho W + \Sigma_Z{}^Y \partial_Y \partial_\rho W\right) = 0\,. \end{aligned} \tag{2.21}$$

[1] In deriving (2.16) we set $\gamma \approx 1$. This is consistent as long as e^{-2U} stays finite. In the infrared, however, e^{-2U} diverges in which case the approximation $\gamma \approx 1$ breakes down. Thus, (2.16) cannot be trusted in the infrared.

Making again use of the curved BPS flow equations we find that to order Λ^2 (2.21) contains terms proportional to f, Λ, f^2 and to Λ^2. All these terms precisely cancel out by virtue of the relations (2.16) and (2.18)!

Next, let us turn to curved BPS flow equations for the warp factor and for the scalar fields. These are given by [8]

$$\begin{aligned} U' &= \gamma g W \,, \\ \rho' &= -\frac{g}{4}\gamma^{-1}\rho^2\partial_\rho W \,, \\ q^{X\prime} &= 3g g^{XY}\Sigma_Y{}^Z\partial_Z W \,, \end{aligned} \tag{2.22}$$

where, again, we chose the upper sign. We find that (2.9) is a solution to (2.22) to all orders in Λ.

Let us first solve (2.22) to linear order in the cosmological constant Λ. To this order, the BPS flow equation for f is given by

$$\begin{aligned} f' &= 3g\left(\sqrt{1-\gamma^2}\sqrt{\frac{1-V}{V}}\frac{(\rho^6-2)}{\rho^2} - f\rho^4\frac{(4V+\rho^6-2)}{2+\rho^6(2V-1)}\right) \\ &= 2\Lambda \mathrm{e}^{-U_0}\sqrt{\frac{V_0}{1-V_0}}\left(-5+4V_0+\rho_0^6\right) , \end{aligned} \tag{2.23}$$

where we used (2.16) to rewrite the r.h.s.

On the the other hand, we can also compute f' directly from (2.16). Using the flat BPS equations

$$\begin{aligned} U_0' &= g\frac{(2+\rho_0^6(2V_0-1))}{2V_0\rho_0^2} \,, \\ V_0' &= 3g\frac{(V_0-1)(\rho_0^6-2)}{\rho_0^2} \,, \\ \rho_0' &= g\frac{[1+\rho_0^6(1-2V_0)]}{2V_0\rho_0} \,, \end{aligned} \tag{2.24}$$

we find that the expression for f' computed from (2.16) is identical to (2.23)! That is, to lowest order in the cosmological constant, the BPS flow equation for f is solved by (2.16).

Next, let us solve the curved BPS flow equations for the fields U, V and ρ to linear order in Λ. These are given by

$$\begin{aligned} U' &= g\left(\frac{2+\rho^6(2V-1)}{2V\rho^2} + f\frac{(2-\rho^6)}{2V\rho^2}\right) , \\ V' &= \frac{3g}{\rho^2}\Bigg(\left(V-1+\sqrt{1-\gamma^2}\sqrt{1-V}\sqrt{V}\right)(\rho^6-2) - \\ &\qquad - f\frac{(-4+\rho^6(4-\rho^6+V(-6+4V+3\rho^6)))}{2+(-1+2V)\rho^6}\Bigg) , \\ \rho' &= g\left(\frac{2+\rho^6(2V-1)(-1+\rho^6(1-2V))}{2V\rho(2+\rho^6(-1+2V))} + f\frac{(1+\rho^6)}{2V\rho}\right) , \end{aligned} \tag{2.25}$$

where f is given by (2.16). These flow equations are quite complicated. In the following we restrict ourselves to solving them in the vicinity of the UV fixed point. We set

$$g = \frac{2}{3} \tag{2.26}$$

to make contact with the results of [5]. Near the UV fixed point, $gW = 1$ and hence $U = r$ as well as $\sqrt{1-\gamma^2} = 2\Lambda e^{-r}$. We will need the explicit form of the flat domain wall solution in the following, which is given by [5]

$$\begin{aligned} V_0 &= 1 - \delta V_0\,, \qquad \delta V_0 = a_0^2\, e^{-2r}\,, \\ \rho_0 &= 1 + \delta\rho_0\,, \qquad \delta\rho_0 = \frac{2}{3} a_0^2 e^{-2r} r + \frac{a_1}{\sqrt{6}} e^{-2r}\,. \end{aligned} \tag{2.27}$$

By inserting (2.27) into (2.16), we find that

$$f = 2\Lambda\, a_0\, e^{-2r} \left(1 - \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2} - 2r\right) \tag{2.28}$$

near the UV fixed point.

Near the UV fixed point, we infer from (2.25) that the curved flow equations for V and ρ are, to lowest order in Λ, given by

$$\begin{aligned} (\delta V)' &= -2\delta V + 4\Lambda e^{-r}\sqrt{\delta V_0}\,, \\ (\delta\rho)' &= \frac{2}{3}(\delta V + f) - 2\delta\rho\,, \end{aligned} \tag{2.29}$$

where, as before,

$$\begin{aligned} V &= 1 - \delta V\,, \\ \rho &= 1 + \delta\rho\,. \end{aligned} \tag{2.30}$$

The eqs. (2.29) are solved by

$$\begin{aligned} \delta V &= \left(a_0^2 + \Delta_V(\Lambda) + 4\Lambda a_0 r\right) e^{-2r}\,, \\ \delta\rho &= \left(\frac{2}{3}a_0^2 r + \frac{a_1}{\sqrt{6}}\right) e^{-2r} + \frac{4}{3}\Lambda\, a_0 \left(1 - \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2}\right) r e^{-2r} + \\ &\quad + \left(\frac{2}{3}\Delta_V(\Lambda) r + \Delta_\rho(\Lambda)\right) e^{-2r}\,, \end{aligned} \tag{2.31}$$

where $\Delta_V(\Lambda)$ and $\Delta_\rho(\Lambda)$ denote integration constants satisfying $\Delta_V = \Delta_\rho = 0$ for $\Lambda = 0$, which will be determined below.

Let us now solve the BPS flow eqs. (2.22) to order Λ^2. To order Λ^2 and to second order in f, the BPS flow equation for f is given by

$$\begin{aligned} f' = 2 \Bigg(&\sqrt{1-\gamma^2}\sqrt{\frac{1-V}{V}}\frac{(\rho^6 - 2)}{\rho^2} - f\rho^4 \frac{(4V + \rho^6 - 2)}{2 + \rho^6(2V-1)} - \sqrt{1-\gamma^2} f \frac{(-3+2V)(-2+\rho^6)}{2\sqrt{(1-V)V}\rho^2} - \\ &- f^2 \frac{\rho_0^4[-4 - 8V_0 + 4(1 + 2(-1+V_0)V_0)\rho_0^6 + (-1+6V_0)\rho_0^{12}]}{2(2 + (-1+2V_0)\rho_0^6)^2} \Bigg)\,. \end{aligned} \tag{2.32}$$

Here V and ρ denote the order Λ-corrected expressions satisfying (2.25), whereas V_0 and ρ_0 denote the expressions satisfying the flat BPS eqs. (2.24). The expression for f, on the other hand, is given by (2.18).

f' can also be computed directly from (2.18). By using the BPS eqs. (2.24) and (2.25) we find that the expression for f' computed from (2.18) is identical to (2.32)! Thus, to quadratic order in the cosmological constant, the BPS eq. (2.32) for f is solved by (2.18).

To second order in f and in Λ, the BPS flow equation for U, V and ρ are given by

$$\begin{aligned}
U' &= \gamma\frac{2+\rho^6(2V-1)}{3V\rho^2} + f\frac{(2-\rho^6)}{3V\rho^2} + f^2\frac{\rho^4(-2+4V+\rho^6)}{6V(2+\rho^6(2V-1))}\,,\\
V' &= \frac{2}{\rho^2}\left[\gamma(V-1)+\sqrt{1-\gamma^2}\sqrt{1-V}\sqrt{V}\right](\rho^6-2) - \\
&\quad - 2f\frac{[-4+\rho^6(4-\rho^6+V(-6+4V+3\rho^6))]}{\rho^2(2+(-1+2V)\rho^6)} - \\
&\quad - f\sqrt{1-\gamma^2}\frac{\sqrt{V}[4-8(-1+V)^2\rho^6+(3-4V)\rho^{12}]}{\sqrt{1-V}\rho^2(2+(-1+2V)\rho^6)} - \\
&\quad - f^2\frac{\rho^4[4-24V^2-4(1+2(-1+V)V)\rho^6+(1-4V+6V^2)\rho^{12}]}{(2+(-1+2V)\rho^6)^2}\,,\\
\rho' &= \frac{1+\rho^6-2V\rho^6}{3\gamma V\rho} + f\frac{(1+\rho^6)}{3V\rho} + f^2\frac{\rho^5[4V^2\rho^6+(-2+\rho^6)^2-2V(4+2\rho^6+\rho^{12})]}{6V(2+(-1+2V)\rho^6)^2}\,.
\end{aligned} \tag{2.33}$$

As before, we restrict ourselves to solving these BPS equations in the vicinity of the UV fixed point, where

$$U = r + \Lambda^2 e^{-2r}\,, \qquad V = 1-\delta V\,, \qquad \rho = 1+\delta\rho\,, \qquad \sqrt{1-\gamma^2} = 2\Lambda e^{-r}\,. \tag{2.34}$$

Near the UV fixed point $\delta V, \delta\rho$ and f are, to leading order, proportional to e^{-2r}. To order e^{-2r}, the BPS eqs. (2.33) reduce to

$$\begin{aligned}
(\delta V)' &= -2\delta V + 4\Lambda e^{-r}\sqrt{\delta V} + 2\Lambda e^{-r}\frac{f}{\sqrt{\delta V}}\,,\\
(\delta\rho)' &= \frac{2}{3}(\delta V + f) - 2\delta\rho\,.
\end{aligned} \tag{2.35}$$

From (2.18), on the other hand, we infer that, to order e^{-2r}, f is given by

$$f = \Lambda e^{-r}\left(2\sqrt{\delta V} - 6\frac{\delta\rho}{\sqrt{\delta V}}\right) + \Lambda^2 e^{-2r}\left(2 - 18\left(\frac{\delta\rho_0}{\delta V_0}\right)^2\right)\,, \tag{2.36}$$

where δV_0 and $\delta\rho_0$ are given by (2.27). Inserting (2.36) into (2.35) yields

$$\begin{aligned}
(\delta V)' &= -2\delta V + 4\Lambda e^{-r}\sqrt{\delta V} + 4\Lambda^2 e^{-2r}\left(1 - 3\frac{\delta\rho}{\delta V}\right) + \mathcal{O}(\Lambda^3)\,,\\
(\delta\rho)' &= \frac{2}{3}\delta V - 2\delta\rho + \frac{4}{3}\Lambda e^{-r}\left(\sqrt{\delta V} - 3\frac{\delta\rho}{\sqrt{\delta V}}\right) + \frac{4}{3}\Lambda^2 e^{-2r}\left(1 - 9\left(\frac{\delta\rho_0}{\delta V_0}\right)^2\right)\,.
\end{aligned} \tag{2.37}$$

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The eqs. (2.37) are solved by

$$\begin{aligned}
\delta V &= \Bigg(a_0^2 + \Sigma^{(1)} + \Sigma^{(2)} - 2\Lambda a_0 \left(1 - \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2}\right) - \\
&\quad - \frac{\Lambda \Sigma^{(1)}}{a_0}\left(1 + \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2}\right) + 6\frac{\Lambda \Delta_\rho}{a_0} + 4\Lambda a_0 \left(1 + \frac{\Sigma^{(1)}}{2a_0^2}\right) r\Bigg)\, \mathrm{e}^{-2r}\,, \\
\delta\rho &= \left(\frac{2}{3}\Big(a_0^2 + \Sigma^{(1)} + \Sigma^{(2)}\Big) r + \frac{a_1}{\sqrt{6}} + \Delta_\rho\right) \mathrm{e}^{-2r}\,,
\end{aligned} \tag{2.38}$$

where $\Sigma^{(1)}$ and $\Sigma^{(2)}$ denote two integration constants of order Λ and Λ^2, respectively, and where Δ_ρ denotes a third integration constant of order Λ and higher. Below we will determine the integration constants $\Sigma^{(1)}$ and $\Sigma^{(2)}$ by comparison with the dual field theory. The presence of Δ_ρ, on the other hand, is crucial in order to obtain invariants under rescalings of r, as we will discuss below.

By setting $\Delta_V = \Sigma^{(1)} - 2\Lambda\, a_0 (1 - \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2})$ we note that (2.38) reduces to (2.31) to linear order in Λ.

Inserting (2.38) into (2.36) yields

$$f = \left(2\Lambda a_0 \left(1 - \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2}\right) + \frac{\Lambda\Sigma^{(1)}}{a_0}\left(1 + \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2}\right) - 6\frac{\Lambda\Delta_\rho}{a_0} - 4\Lambda a_0 \left(1 + \frac{\Sigma^{(1)}}{2a_0^2}\right) r\right) \mathrm{e}^{-2r}\,. \tag{2.39}$$

Then we compute

$$f + \delta V = \left(a_0^2 + \Sigma^{(1)} + \Sigma^{(2)}\right) \mathrm{e}^{-2r}\,. \tag{2.40}$$

Using (2.10) we obtain

$$\tau = a_0 \left(1 + \frac{\Sigma^{(1)}}{2a_0^2} - \frac{(\Sigma^{(1)})^2}{8a_0^4} + \frac{\Sigma^{(2)}}{2a_0^2}\right) \mathrm{e}^{-r}\,. \tag{2.41}$$

If we demand that there are no terms proportional to Λ^2 in (2.41), then we infer that

$$\Sigma^{(2)} = \frac{(\Sigma^{(1)})^2}{4a_0^2}\,. \tag{2.42}$$

In the next section we will discuss the dual field theory realisation yielding (2.42).

Using (2.42), we thus find that to order Λ^2, and near the UV fixed point $gW = 1$, the curved domain wall solution is supported by the three scalar fields

$$\begin{aligned}
\rho &= 1 + \left(\frac{2}{3}a_0^2\left(1 + \frac{\Sigma^{(1)}}{2a_0^2}\right)^2 r + \frac{a_1}{\sqrt{6}} + \Delta_\rho\right)\mathrm{e}^{-2r}\,, \\
f &= \left(-4\Lambda a_0 \left(1 + \frac{\Sigma^{(1)}}{2a_0^2}\right) r + 2\Lambda a_0 \left(1 - \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2}\right) + \frac{\Lambda\Sigma^{(1)}}{a_0}\left(1 + \sqrt{\frac{3}{2}}\frac{a_1}{a_0^2}\right) - 6\frac{\Lambda\Delta_\rho}{a_0}\right)\mathrm{e}^{-2r}\,, \\
\tau &= a_0\left(1 + \frac{\Sigma^{(1)}}{2a_0^2}\right)\mathrm{e}^{-r}\,.
\end{aligned} \tag{2.43}$$

We note that in deriving (2.43) we worked to order e^{-2r}.

The presence of the integration constant a_1 in the flat domain wall solution (2.27) is necessary for the solution to be invariant under additive shifts of r ($r \to r + \beta$) [5]. Under such an additive shift, a_0 and a_1 transform as $a_0 \to \mathrm{e}^{\beta} a_0$, $a_1 \to \mathrm{e}^{\beta}(a_1 - \sqrt{\frac{8}{3}} a_0^2 \beta)$. The invariant combination is thus given by $a_1/a_0^2 + \sqrt{\frac{8}{3}} \log a_0$, and its value is determined by demanding that the RG flow terminates at a superconformal fixed point in the IR [5]. In the curved case, on the other hand, we have $\Lambda \to \mathrm{e}^{\beta}\Lambda$, $\Sigma^{(1)} \to \mathrm{e}^{2\beta}\Sigma^{(1)}$ under additive shifts of r. Then, to order Λ^2, the solution (2.43) is invariant under these additive shifts provided that $\Delta_\rho \to \mathrm{e}^{2\beta}[\Delta_\rho - \frac{2}{3}(\Sigma^{(1)} + \frac{1}{4}(\Sigma^{(1)})^2/a_0^2)\beta]$.

The three non-constant scalar fields ρ, f and τ should be in one-to-one correspondence with the deformations in the dual field theory. $\delta\rho$ and f, which contain terms proportional to $r\mathrm{e}^{-2r}$, should correspond to bosonic mass deformations, whereas τ, which behaves as e^{-r}, should correspond to a fermionic mass deformation [3, 15]. We will, in the next section, show that this is indeed the case. In doing so, we will determine the integration constant $\Sigma^{(1)}$.

3. The dual field theory on a curved background

The dual field theory studied by Freedman, Gubser, Pilch and Warner consists of $\mathcal{N} = 4$ super-Yang-Mills theory broken to an $\mathcal{N} = 1$ theory by the addition of a mass term for one of the three adjoint chiral superfields. This field theory may be put on a curved background by setting the gravitational superfield $\mathcal{R}$ to a constant value and by setting $W_{\alpha\beta\gamma} = G_{\alpha\dot{\alpha}} = 0$ (see [16] and references therein). This implies that i) the auxiliary field b_a vanishes, ii) the lowest component field M of $\mathcal{R}$ acquires a constant value, i.e. $M = 6\Lambda \mathrm{e}^{i\alpha}$ and iii) the curvature scalar R becomes constant, $R = \frac{4}{3}\bar{M}M = 48\Lambda^2$ (we use the notation of [13]).

The superfield lagrangian describing the dual field theory on a curved background is, in $\mathcal{N} = 1$ superspace notation [13], given by[2]

$$\mathcal{L} = \int d^2\Theta 2\varepsilon \left[\frac{1}{4k} \operatorname{Tr} W^\alpha W_\alpha - \frac{1}{8}(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}} - 8\mathcal{R})\Phi^\dagger \mathrm{e}^V \Phi + \frac{m}{2}\operatorname{Tr}\Phi^2 + c_1 \mathcal{R} \operatorname{Tr}\Phi^2 \right] + \text{h.c.}\,. \tag{3.1}$$

Note that we have allowed for the presence of an additional holomorphic term of the form $\mathcal{R}\operatorname{Tr}\Phi^2$, with an unspecified dimensionless coefficient c_1!

The component expansion of (3.1) gives rise to a fermionic mass term

$$\left(m - \frac{c_1}{3}M\right)\chi\chi\,. \tag{3.2}$$

It also gives rise to the following auxiliary field contributions,

$$\begin{aligned} \frac{\mathcal{L}_{\text{aux}}}{e} = & \frac{1}{9}\bar{A}A\left|M - 3\frac{\bar{F}}{\bar{A}}\right|^2 - \frac{1}{2}\left(m - \frac{c_1}{3}M\right)\bar{M}A^2 - \frac{1}{2}\left(m - \frac{c_1}{3}\bar{M}\right)M\bar{A}^2 + \\ & + \left(m - \frac{c_1}{3}M\right)AF + \left(m - \frac{c_1}{3}\bar{M}\right)\bar{A}\bar{F}\,. \end{aligned} \tag{3.3}$$

[2]There is also a cubic term $\operatorname{Tr}([\Phi_1, \Phi_2]\Phi_3)$ in the superpotential W, which we will omit in the following.

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Therefore, the equation of motion for the auxiliary field F reads

$$\bar{F} = -\left(m - \frac{c_1}{3}M\right) A + \frac{1}{3} M \bar{A} \,. \tag{3.4}$$

Reinserting (3.4) into (3.3) yields

$$\frac{\mathcal{L}_{\text{aux}}}{e} = -\left(\left|m - \frac{c_1}{3}M\right|^2 \bar{A}A + \frac{1}{6}\left(m - \frac{c_1}{3}M\right)\bar{M}A^2 + \frac{1}{6}\left(m - \frac{c_1}{3}\bar{M}\right) M\bar{A}^2\right). \tag{3.5}$$

The kinetic part $\mathcal{L}_{\text{kin}}$ of the lagrangian (3.1), on the other hand, contains a term [13]

$$\frac{\mathcal{L}_{\text{kin}}}{e} = \frac{1}{6} R\bar{A}A + \cdots = \frac{2}{9}\bar{M}M\bar{A}A + \cdots \,. \tag{3.6}$$

Thus, the resulting effective scalar potential is given by

$$V_{\text{eff}} = \left(\left|m - \frac{c_1}{3}M\right|^2 - \frac{2}{9}\bar{M}M\right)\bar{A}A + \frac{1}{6}\left(m - \frac{c_1}{3}M\right)\bar{M}A^2 + \frac{1}{6}\left(m - \frac{c_1}{3}\bar{M}\right) M\bar{A}^2 \,. \tag{3.7}$$

In the following, we will only consider the case when $M = \bar{M} = 6\Lambda$, i.e. $\alpha = 0$. Then we obtain

$$V_{\text{eff}} = \left(m^2 - 4c_1 m\Lambda + 4(c_1^2 - 2)\Lambda^2\right)\bar{A}A + \Lambda(m - 2c_1\Lambda)(A^2 + \bar{A}^2) \,. \tag{3.8}$$

Let us consider what happens when we set $m = c_1 = 0$. Then the lagrangian (3.1) describes $\mathcal{N} = 4$ super-Yang-Mills theory on an AdS_4 background, and we do not expect to have an RG flow. The associated dual curved gravitational solution has constant scalars ($gW = 1$). The curved BPS equation for the warp factor now reads (cf. (2.22)) $U' = \gamma = 1 - 2\Lambda^2 \mathrm{e}^{-2U} + \mathcal{O}(\Lambda^4)$, which is solved by $\mathrm{e}^U = \mathrm{e}^r + \Lambda^2 \mathrm{e}^{-r}$. This corresponds to a curved slicing of AdS_5 [14]. Thus, the effect of the term $-8\Lambda^2\bar{A}A$ in V_{eff} is to induce a modification of the warp factor U.

On the other hand, when turning on either m and/or c_1, we do induce an RG flow. We therefore expect to have a correspondence between the non-constant scalar fields (2.43) supporting the curved domain wall and the fermionic and the bosonic deformations given in (3.2) and in

$$\tilde{V}_{\text{eff}} = V_{\text{eff}} + 8\Lambda^2\bar{A}A = (m - 2c_1\Lambda)^2\,\bar{A}A + \Lambda(m - 2c_1\Lambda)(A^2 + \bar{A}^2) \,, \tag{3.9}$$

respectively. We will, in the following, make the identification $a_0 = m$ [5]. Inspection of (2.43) then shows that $\delta\rho$ and τ are in correspondence [5] with the operators $\bar{A}A$ and $\chi\chi$, respectively, whereas f is in correspondence with the operator A^2. Note that the latter gets switched off when turning off the cosmological constant Λ.

Let us have a closer look at the correspondence between τ and $\chi\chi$. The identification of τ given in (2.43) with (3.2) yields

$$\Sigma^{(1)} = -4c_1\, a_0\, \Lambda \,. \tag{3.10}$$

Next, let us consider the correspondence of f with A^2. The coefficient multiplying the term $r\mathrm{e}^{-2r}$ in f is proportional to $\Lambda(a_0 + \frac{1}{2}\Sigma^{(1)}/a_0) = \Lambda(a_0 - 2c_1\Lambda)$. This is precisely the coefficient of the A^2 term in (3.9)!

And finally, let us consider the correspondence of $\delta\rho$ with $\bar{A}A$. The coefficient multiplying the term re^{-2r} in $\delta\rho$ is proportional to $(a_0 + \frac{1}{2}\Sigma^{(1)}/a_0)^2 = (a_0 - 2c_1\Lambda)^2$. This too is in precise agreement with the coefficient of the $\bar{A}A$ term in (3.9)!

We may add further perturbations to (3.1), for instance couplings of the form

$$\sum_{p\geq 2}^{\infty} c_p m^{1-p} \int d^2\Theta 2\varepsilon \mathcal{R}^p \,\mathrm{Tr}\,\Phi^2 + \text{h.c.} \tag{3.11}$$

with dimensionless coefficients c_p. We expect that these can also be captured by the curved domain wall solution. Consider, for instance, adding $\mathcal{R}^2\,\mathrm{Tr}\,\Phi^2$ $(p = 2)$. This will result in additional Λ^2 corrections to $\chi\chi$ and to $\bar{A}A$, which can be captured on the domain wall side by replacing (2.42) by

$$\Sigma^{(2)} = \frac{(\Sigma^{(1)})^2}{4a_0^2} + 4c_2\Lambda^2 = 4\left(c_1^2 + c_2\right)\Lambda^2\,. \tag{3.12}$$

To summarise, we have found that the dictionary between the scalar fields supporting the curved domain wall solution and the deformations in the dual field theory on an AdS_4 background is given by

$$\begin{aligned} \delta\rho &\longleftrightarrow \bar{A}A\,, \\ f &\longleftrightarrow A^2\,, \\ \tau &\longleftrightarrow \chi\chi\,. \end{aligned} \tag{3.13}$$

4. Conclusions

In this note we put the field theory studied by Freedman, Gubser, Pilch and Warner on a curved AdS_4 background, and we constructed the curved supersymmetric domain wall solution which describes this field theory near the UV fixed point. In doing so we allowed for the presence of additional deformations in the field theory, for instance of $\mathcal{R}\,\mathrm{Tr}\,\Phi^2$. This example of a curved BPS domain wall demonstrates that holographic RG flows in supersymmetric field theories on a curved AdS_4 background can be described in terms of curved BPS domain wall solutions. We note that these curved domain wall solutions are distinct from the AdS_4 brane solution discussed in [17]–[20], describing the embedding of a probe $D5$-brane in AdS_5. The dual field theory description of the AdS_4 brane solution of [17]–[20] is in terms of a defect field theory on a flat background, whereas the dual field theory we considered in this note lives on an AdS_4 background.

We constructed the curved domain wall solution to order Λ^2. Inspection of (2.8) shows that the curved domain wall solution will receive further corrections which are higher order in Λ (i.e. of order Λ^3, Λ^4, etc.). These will become important when flowing towards the infrared. It would be interesting to investigate this further.

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